AP Test Review

Name: ______________________

Date: ________________ Period: ______

Question Type #7- Implicit Differentiation and Related Rates

What you should be able to do:

- Implicit Differentiation: Interpret a derivative in $\frac{dy}{dx}$ notation as the rate of change of one quantity with respect to another and thus indicative of a dependent and independent variable.
- Read, understand, and translate the verbal description into symbols relating the variables that are changing.
- Understand what quantities are changing and which are constant throughout the problem.
- Understand the geometry of the situation. Use geometric relationships to replace variables with another. This may require working with similar figures.
- Do implicit differentiation with respect to time.
- A common misunderstanding is to substitute specific values of variables into the equation before differentiating:
  - o Any quantity that does not change during the course of the problem can be substituted at the beginning
  - o Any quantity that changes should not be substituted until the derivative is being evaluated at the specific time stated in the question

Multiple Choice Practice: Calculator

1. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the crossing and watches an eastbound train traveling at 60 meters per second. At how many meters per second is the train moving away from the observer 4 seconds after it passes through the intersection?

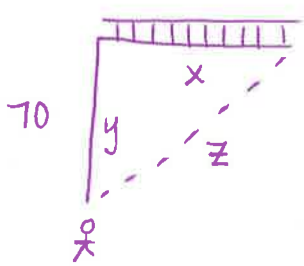
a) 57.60

b) 57.88

c) 59.20

d) 60.00

e) 67.40



$\frac{dx}{dt} = 60$ m/s $\qquad \frac{dz}{dt} = ?$ $\qquad x = 4 \cdot 60$

$x = 240$

constant

$x^2 + 70^2 = z^2$ $\qquad 70^2 + 240^2 = z^2$

$z = 250$

$2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$

$2(240)(60) = 2(250)\frac{dz}{dt}$

$28800 = 500\frac{dz}{dt}$

2. In the triangle shown above, if θ increases at a constant rate of 3 radians per minute, at what rate is x increasing in units per minute when x equals 3 units?

$\frac{d\theta}{dt} = 3$

a) 3

b) $\frac{15}{4}$

c) 4

d) 9

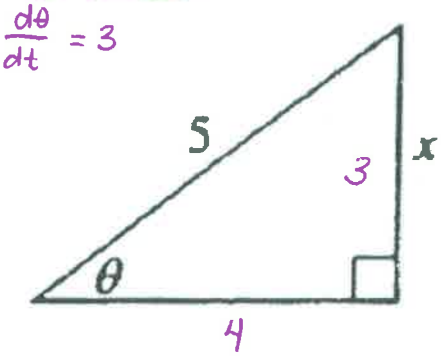
e) 12

$\sin\theta = \frac{x}{5}$

$\cos\theta\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$

$\left(\frac{4}{5}\right)(3) = \frac{1}{5}\frac{dx}{dt}$

$\frac{12}{5} = \frac{1}{5}\frac{dx}{dt}$



3. The radius of a circle is decreasing at a constant rate of 0.1 centimeters per second. In terms of the circumference C, what is the rate of change of the area of the circle, in square centimeters per second?

a) $-(0.2)\pi C$

b) $-(0.1)C$

c) $-\frac{(0.1)C}{2\pi}$

d) $(0.1)^2 C$

e) $(0.1)^2 \pi C$

$\frac{dr}{dt} = -0.1 \text{ cm/s}$ $\quad \frac{dA}{dt} = ?$

$A = \pi r^2$

$\frac{dA}{dt} = \underbrace{2\pi r}\, \frac{dr}{dt}$

$\frac{dA}{dt} = C(-0.1)$

Multiple Choice Practice: Non-Calculator

4. If the base b of a triangle is increasing at a rate of 3 inches per minute while its height h is decreasing at a rate of 3 inches per minute, which of the following must be true about the area A of the triangle?

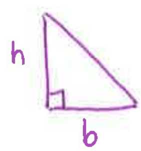
a) A is always increasing

b) A is always decreasing

c) A is decreasing only when $b < h$

d) A is decreasing only when $b > h$

e) A remains constant



$\frac{db}{dt} = 3 \quad \frac{dh}{dt} = -3$

$A = \frac{1}{2} b \cdot h$

$\frac{dA}{dt} = \frac{1}{2}\left(\frac{db}{dt} h + \frac{dh}{dt} b\right)$

$\frac{dA}{dt} = \frac{1}{2}(3h - 3b)$

$\frac{dA}{dt} = \frac{3}{2}(h - b)$

5. The radius of a circle is increasing at a constant rate of 0.2 meters per second. What is the rate of increase in the area of the circle at the instant when the circumference of the circle is 20π meters?

a) $0.04\pi\ m^2/sec$

b) $0.4\pi\ m^2/sec$

c) $4\pi\ m^2/sec$

d) $20\pi\ m^2/sec$

e) $100\pi\ m^2/sec$

$\frac{dr}{dt} = 0.2 \text{ m/s} \quad \frac{dA}{dt} = ?$ $\quad C = 20\pi$ $\quad C = 2\pi r$

$20\pi = 2\pi r$

$r = 10$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(10)(0.2)$

6. The slope of the line tangent to the curve $y^2 + (xy+1)^3 = 0$ at $(2, -1)$ is

a) $-\frac{3}{2}$

b) $-\frac{3}{4}$

c) 0

d) $\frac{3}{4}$

e) $\frac{3}{2}$

$2y\frac{dy}{dx} + 3(xy+1)^2\left(\frac{dx}{dx}y + \frac{dy}{dx}x + 0\right) = 0$

$2(-1)\frac{dy}{dx} + 3((2)(-1)+1)^2\left((-1) + 2\frac{dy}{dx}\right) = 0$

$-2\frac{dy}{dx} + 3(-1)^2\left(-1 + 2\frac{dy}{dx}\right) = 0$

$-2\frac{dy}{dx} + 3\left(-1 + 2\frac{dy}{dx}\right) = 0$

$-2\frac{dy}{dx} - 3 + 6\frac{dy}{dx} = 0$

$4\frac{dy}{dx} = 3$

$\frac{dy}{dx} = 3/4$

7. If $\frac{dy}{dx} = \sqrt{1-y^2}$, then $\frac{d^2y}{dx^2} =$

a) $-2y$

b) $-y$

c) $-\frac{y}{\sqrt{1-y^2}}$

d) y

e) $1/2$

$$\frac{d^2y}{dx^2} = \frac{1}{2}(1-y^2)^{-1/2} \cdot -2y\frac{dy}{dx}$$

$$= \frac{-\cancel{2}y}{\cancel{2}\cancel{\sqrt{1-y^2}}} \cdot \cancel{\sqrt{1-y^2}}$$

$$= -y$$

8. If $x^2 + y^2 = 25$, what is the value of $\frac{d^2y}{dx^2}$ at the point (4, 3)?

a) $-\frac{25}{27}$

b) $-\frac{7}{27}$

c) $\frac{7}{27}$

d) $\frac{3}{4}$

e) $\frac{25}{27}$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} \text{ at } (4,3) = \frac{-4}{3}$$

$$\frac{d^2y}{dx^2} = \frac{(-1)(y) - \frac{dy}{dx}(-x)}{y^2}$$

$$\frac{d^2y}{dx^2} \text{ at } (4,3) = \frac{(-1)(3) - (-\frac{4}{3})(-4)}{3^2}$$

$$= \frac{-3 - \frac{16}{3}}{9} = \frac{\frac{-25}{3}}{9}$$

9. If $y = xy + x^2 + 1$, then when x = -1, $\frac{dy}{dx}$ is

a) $\frac{1}{2}$

b) $-\frac{1}{2}$

c) -1

d) -2

e) *nonexistent*

$$\frac{dy}{dx} = (1)(y) + \frac{dy}{dx}x + 2x\frac{dx}{dx} + 0$$

$$\frac{dy}{dx} = y + x\frac{dy}{dx} + 2x$$

$$\frac{dy}{dx} = (1) + (-1)\frac{dy}{dx} + 2(-1)$$

$$\frac{dy}{dx} = -1 - \frac{dy}{dx}$$

$$2\frac{dy}{dx} = -1 \qquad \frac{dy}{dx} = -1/2$$

$$y = (-1)y + (-1)^2 + 1$$

$$y = -1y + 1 + 1$$

$$2y = 2$$

$$y = 1$$

ANSWERS: 1. A 2. E 3. B 4. D 5. C 6. D 7. B 8. A 9. B

2008 AP Test Free Response #6 (FORM B)

NON-CALCULATOR

Consider the closed curve in the xy-plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x-axis? Explain your reasoning.

A) $2x+2+4y^3\frac{dy}{dx}+4\frac{dy}{dx}=0$

$4y^3\frac{dy}{dx}+4\frac{dy}{dx}=-2x-2$

$(4y^3+4)\frac{dy}{dx}=-2x-2$

$\frac{dy}{dx}=\frac{-2(x-1)}{4(y^3+1)} \rightarrow \boxed{\frac{-(x+1)}{2(y^3+1)}}$

B) $\left.\frac{dy}{dx}\right|_{(-2,1)} \frac{-(-2+1)}{2(1+1)}=\frac{1}{4} \rightarrow \boxed{y-1=\frac{1}{4}(x+2)}$

c) vertical tangents exist when $\frac{dy}{dx}$ is undefined.

Let $2(y^3+1)=0$, $y=-1$ & $x \neq -1$; therefore: $x^2+2x+(-1)^4+4(-1)=5$

$x^2+2x-8=0$

$(x+4)(x-2)=0 \quad x=-4,2$

Two tangent points: $\boxed{(-4,-1) \ \& \ (2,-1)}$

d) Horizontal tangents: when $dy/dx=0$.

Let $-(x+1)=0$, $x=-1$ & $y \neq -1$. This is where horizontal tangents lie. x-intercepts lie when $y=0$. Let $(-1)^2+2(-1)+(0)^4+4(0)=5 \rightarrow$ Not true. This never happens!

Question 6

Consider the closed curve in the xy-plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{-(x+1)}{2(y^3+1)}$.

(b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.

(c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.

(d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x-axis? Explain your reasoning.

(a) $2x + 2 + 4y^3\dfrac{dy}{dx} + 4\dfrac{dy}{dx} = 0$ $(4y^3+4)\dfrac{dy}{dx} = -2x - 2$ $\dfrac{dy}{dx} = \dfrac{-2(x+1)}{4(y^3+1)} = \dfrac{-(x+1)}{2(y^3+1)}$	2 : { 1 : implicit differentiation 1 : verification
(b) $\left.\dfrac{dy}{dx}\right\vert_{(-2,1)} = \dfrac{-(-2+1)}{2(1+1)} = \dfrac{1}{4}$ Tangent line: $y = 1 + \dfrac{1}{4}(x+2)$	2 : { 1 : slope 1 : tangent line equation
(c) Vertical tangent lines occur at points on the curve where $y^3 + 1 = 0$ (or $y = -1$) and $x \neq -1$. On the curve, $y = -1$ implies that $x^2 + 2x + 1 - 4 = 5$, so $x = -4$ or $x = 2$. Vertical tangent lines occur at the points $(-4, -1)$ and $(2, -1)$.	3 : { 1 : $y = -1$ 1 : substitutes $y = -1$ into the equation of the curve 1 : answer
(d) Horizontal tangents occur at points on the curve where $x = -1$ and $y \neq -1$. The curve crosses the x-axis where $y = 0$. $(-1)^2 + 2(-1) + 0^4 + 4 \cdot 0 \neq 5$ No, the curve cannot have a horizontal tangent where it crosses the x-axis.	2 : { 1 : works with $x = -1$ or $y = 0$ 1 : answer with reason

2005 AP Test Free Response #5 (FORM B)

NON-CALCULATOR

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\frac{dy}{dx} = \frac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\frac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\frac{dy}{dt} = 6$. Find the value of $\frac{dx}{dt}$ at time $t = 5$.

A) $2y\frac{dy}{dx} = y + x\frac{dy}{dx}$

$2y\frac{dy}{dx} - x\frac{dy}{dx} = y$

$\frac{dy}{dx} = \frac{y}{2y-x}$ ✓

B) Let $\frac{y}{2y-x} = \frac{1}{2}$

$2y = 2y - x$

$x = 0$

$y^2 = 2 + xy$

$y^2 = 2$

$y = \pm\sqrt{2}$

$\boxed{(0, \sqrt{2}), (0, -\sqrt{2})}$

c) $\frac{y}{2y-x} = 0$

$y = 0.$

but – if $y^2 = 2 + xy$

$0^2 \neq 2 + x(0)$

No horizontal tangents.

d) when $y = 3$ $\quad 3^2 = 2 + x(3)$

$7 = 3x$

$7/3 = x$

We know $\frac{dy}{dt} = \frac{dy/dx}{dx/dt} = \frac{y}{2y-x} \cdot \frac{dx}{dt}$

At $t = 5$ $\quad 6 = \frac{3}{6 - 7/3} \cdot \frac{dx}{dt} = \frac{9}{11}\frac{dx}{dt}$

$6 = 9/11 \, dx/dt$

$\boxed{dx/dt = 22/3}$

Question 5

Consider the curve given by $y^2 = 2 + xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y}{2y - x}$.

(b) Find all points (x, y) on the curve where the line tangent to the curve has slope $\dfrac{1}{2}$.

(c) Show that there are no points (x, y) on the curve where the line tangent to the curve is horizontal.

(d) Let x and y be functions of time t that are related by the equation $y^2 = 2 + xy$. At time $t = 5$, the value of y is 3 and $\dfrac{dy}{dt} = 6$. Find the value of $\dfrac{dx}{dt}$ at time $t = 5$.

(a) $2yy' = y + xy'$

$(2y - x)y' = y$

$y' = \dfrac{y}{2y - x}$

$2 : \begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{solves for } y' \end{cases}$

(b) $\dfrac{y}{2y - x} = \dfrac{1}{2}$

$2y = 2y - x$

$x = 0$

$y = \pm\sqrt{2}$

$(0, \sqrt{2}), (0, -\sqrt{2})$

$2 : \begin{cases} 1 : \dfrac{y}{2y - x} = \dfrac{1}{2} \\ 1 : \text{answer} \end{cases}$

(c) $\dfrac{y}{2y - x} = 0$

$y = 0$

The curve has no horizontal tangent since $0^2 \neq 2 + x \cdot 0$ for any x.

$2 : \begin{cases} 1 : y = 0 \\ 1 : \text{explanation} \end{cases}$

(d) When $y = 3$, $3^2 = 2 + 3x$ so $x = \dfrac{7}{3}$.

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \dfrac{y}{2y - x} \cdot \dfrac{dx}{dt}$

At $t = 5$, $6 = \dfrac{3}{6 - \frac{7}{3}} \cdot \dfrac{dx}{dt} = \dfrac{9}{11} \cdot \dfrac{dx}{dt}$

$\left.\dfrac{dx}{dt}\right|_{t=5} = \dfrac{22}{3}$

$3 : \begin{cases} 1 : \text{solves for } x \\ 1 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

2004 AP Test Free Response #4

NON-CALCULATOR

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P ? Justify your answer.

A.) $2x + 8y\,dy/dx = 3y + 3x\,dy/dx$

$8y\,dy/dx - 3x\,dy/dx = 3y - 2x$

$dy/dx = \dfrac{3y-2x}{8y-3x}$ ✓

B.) $\dfrac{3y-2x}{8y-3x} = 0$ when $3y - 2x = 0$

when $x = 3$ $\quad 3y - 2(3) = 0$

$3y = 6$

$y = 2$

point $P = (3,2)$ is on the curve & does have slope of zero

$(3)^2 + 4(2)^2 = 7 + 3(3)(2)$

$9 + 16 = 7 + 18$

$25 = 25$ ✓

C.) $\dfrac{dy}{dx} = \dfrac{3y-2x}{8y-3x}\Bigg|_{(3,2)}$ verify: $= \dfrac{6-6}{16-6} = \dfrac{0}{10} = 0$

$\dfrac{d^2y}{dx^2} = \dfrac{(3\,dy/dx - 2)(8y-3x) - (3y-2x)(8\,dy/dx - 3)}{(8y-3x)^2}\Bigg|_{(3,2)} = \dfrac{(3(0)-2)(8(2)-3(3)) - (3(2)-2(3))(8(0)-3)}{(8(2)-3(3))^2}$

$= \dfrac{(-2)(16-9) - (6-6)(-3)}{(16-9)^2}$

$= \dfrac{-2(7)}{7^2} = \dfrac{-14}{49} = \dfrac{-2}{7}$

The curve must have a local max @ point P because $y' = 0$ & $y'' < 0$!

Question 4

Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point P with x-coordinate 3 at which the line tangent to the curve at P is horizontal. Find the y-coordinate of P.

(c) Find the value of $\dfrac{d^2y}{dx^2}$ at the point P found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point P? Justify your answer.

(a)
$$2x + 8yy' = 3y + 3xy'$$
$$(8y - 3x)y' = 3y - 2x$$
$$y' = \frac{3y - 2x}{8y - 3x}$$

$2: \begin{cases} 1: \text{implicit differentiation} \\ 1: \text{solves for } y' \end{cases}$

(b) $\dfrac{3y - 2x}{8y - 3x} = 0;\ 3y - 2x = 0$

When $x = 3,\ 3y = 6$
$y = 2$

$3^2 + 4 \cdot 2^2 = 25$ and $7 + 3 \cdot 3 \cdot 2 = 25$

Therefore, $P = (3, 2)$ is on the curve and the slope is 0 at this point.

$3: \begin{cases} 1: \dfrac{dy}{dx} = 0 \\ 1: \text{shows slope is 0 at } (3, 2) \\ 1: \text{shows } (3, 2) \text{ lies on curve} \end{cases}$

(c) $\dfrac{d^2y}{dx^2} = \dfrac{(8y - 3x)(3y' - 2) - (3y - 2x)(8y' - 3)}{(8y - 3x)^2}$

At $P = (3, 2)$, $\dfrac{d^2y}{dx^2} = \dfrac{(16 - 9)(-2)}{(16 - 9)^2} = -\dfrac{2}{7}$.

Since $y' = 0$ and $y'' < 0$ at P, the curve has a local maximum at P.

$4: \begin{cases} 2: \dfrac{d^2y}{dx^2} \\ 1: \text{value of } \dfrac{d^2y}{dx^2} \text{ at } (3, 2) \\ 1: \text{conclusion with justification} \end{cases}$

2001 AP Test Free Response #6

NON-CALCULATOR

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

A) $\frac{d^2y}{dx^2} = \overbrace{2y\underbrace{\frac{dy}{dx}}_{dy/dx\ sub}(6-2x) + -2y^2}^{(product)}$

$= 2y(y^2(6-2x))(6-2x) - 2y^2$

$\frac{d^2y}{dx^2} = 2y^3(6-2x)^2 - 2y^2 \Big|_{(3, 1/4)}$ $\quad 2\left(\frac{1}{4}\right)^3(6-2(3))^2 - 2\left(\frac{1}{4}\right)^2 = \frac{2}{64}(0)^2 - 2\left(\frac{1}{16}\right)$

$= \boxed{-\frac{1}{8}}$

B.) $\frac{dy}{dx} = y^2(6-2x)$

$dy = y^2(6-2x)dx$

$\frac{1}{y^2}dy = (6-2x)dx$

$\int \frac{1}{y^2}dy = \int (6-2x)dx$

$\frac{-1}{y} = 6x - x^2 + C$

@ $f(3) = 1/4$ $\quad \frac{-1}{1/4} = 6(3) - 3^2 + C$

$-4 = 18 - 9 + C$

$-13 = C$

$\frac{-1}{y} = 6x - x^2 - 13$

$\frac{-1}{6x - x^2 - 13} = y$ or

$\boxed{y = \frac{1}{x^2 - 6x + 13}}$

Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a)
$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6 - 2x) - 2y^2$$
$$= 2y^3(6 - 2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3, \frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

$3 : \begin{cases} 2 : \frac{d^2y}{dx^2} \\ \quad < -2 > \text{ product rule or chain rule error} \\ 1 : \text{value at } \left(3, \frac{1}{4}\right) \end{cases}$

(b)
$$\frac{1}{y^2}\,dy = (6 - 2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

$6 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(3) = \frac{1}{4} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

A.) If $r=5$ when $h=10$ then $r=\frac{1}{2}h$.

So if $h=5$ cm, $r=5/2$ cm

$V=\frac{1}{3}\pi r^2 h$

$V=\frac{1}{3}\pi(5/2)^2(5)$

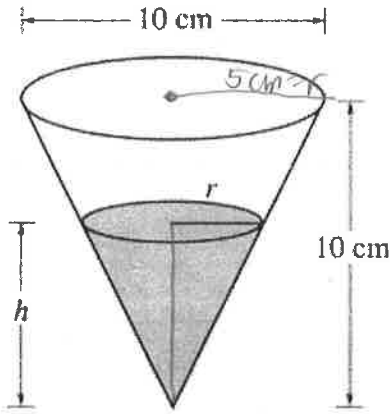
$V=1/3\,\pi(25/4)(5)$

$V=1/3\,\pi(125/4)$

$\boxed{V=\frac{125}{12}\pi \text{ cm}^3/\text{hr}}$

2002 AP Test Free Response Question #5

NON-CALCULATOR



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr. $\frac{dh}{dt}=-3/10$

(Note: The volume of a cone of height h and radius r is given by $V=\frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h=5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h=5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

B.) since $V=\frac{1}{3}\pi r^2 h$ & we know $\frac{dh}{dt}$, eliminate "presence" of r so that $\frac{dr}{dt}$ isn't introduced.

If $r=5$cm when $h=10$cm, then $r=\frac{1}{2}h$. We use this.

Let $V=\frac{1}{3}\pi(\frac{1}{2}h)^2 h \ldots V=\frac{1}{3}\pi(\frac{1}{4}h^3)=V=\frac{1}{12}\pi h^3$

we derive: $dV/dt=\frac{1}{4}\pi h^2\, dh/dt\Big|_{h=5}$ $\frac{1}{4}\pi(5)^2(-3/10)=\frac{-75}{40}=\boxed{\frac{-15}{8}\pi \text{ cm}^3/\text{hr}}$

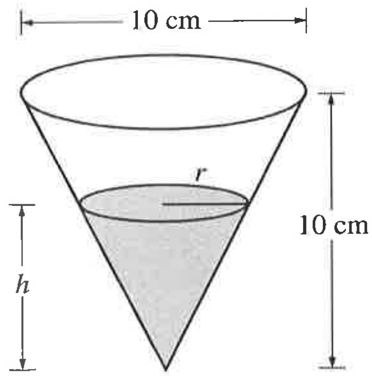
C.) $\frac{dV}{dt}=\frac{1}{4}\pi h^2\frac{dh}{dt}$ (from problem)

$\frac{dV}{dt}=\frac{1}{4}\pi h^2(-3/10)=\frac{dV}{dt}=\frac{-3}{40}\pi h^2$

$=\frac{-3}{40}\pi(2r)^2=\frac{-3}{10}\pi r^2=\frac{-3}{10}(\text{area water})$

constant

Question 5



A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

(a) When $h = 5$, $r = \frac{5}{2}$; $V(5) = \frac{1}{3}\pi\left(\frac{5}{2}\right)^2 5 = \frac{125}{12}\pi \text{ cm}^3$

1 : V when $h = 5$

(b) $\frac{r}{h} = \frac{5}{10}$, so $r = \frac{1}{2}h$

$$V = \frac{1}{3}\pi\left(\frac{1}{4}h^2\right)h = \frac{1}{12}\pi h^3;\ \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5} = \frac{1}{4}\pi(25)\left(-\frac{3}{10}\right) = -\frac{15}{8}\pi \ \text{cm}^3/\text{hr}$$

OR

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right);\ \frac{dr}{dt} = \frac{1}{2}\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{h=5,r=\frac{5}{2}} = \frac{1}{3}\pi\left(\left(\frac{25}{4}\right)\left(-\frac{3}{10}\right) + 2\left(\frac{5}{2}\right)5\left(-\frac{3}{20}\right)\right)$$

$$= -\frac{15}{8}\pi \ \text{cm}^3/\text{hr}$$

5 $\begin{cases} 1:\ r = \frac{1}{2}h \text{ in (a) or (b)} \\ 1:\ \begin{cases} V \text{ as a function of one variable in (a) or (b)} \\ \text{OR} \\ \frac{dr}{dt} \end{cases} \\ 2:\ \frac{dV}{dt} \\ \quad <-2> \text{ chain rule or product rule error} \\ 1:\ \text{evaluation at } h = 5 \end{cases}$

(c) $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} = -\frac{3}{40}\pi h^2$

$$= -\frac{3}{40}\pi(2r)^2 = -\frac{3}{10}\pi r^2 = -\frac{3}{10}\cdot\text{area}$$

The constant of proportionality is $-\frac{3}{10}$.

2 $\begin{cases} 1:\ \text{shows } \frac{dV}{dt} = k\cdot\text{area} \\ 1:\ \text{identifies constant of proportionality} \end{cases}$

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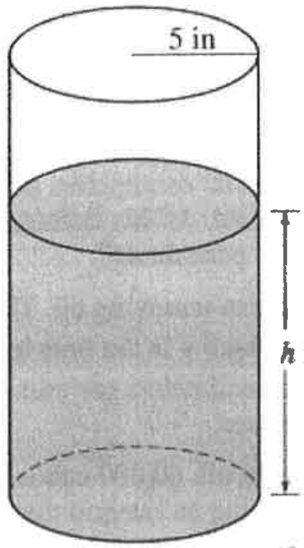
units of cm^3 in (a) and cm^3/hr in (b)

1 : correct units in (a) and (b)

2003 AP Test Free Response Question #5

NON-CALCULATOR



$r=5,\ dr/dt=0$ (no change in radius!)

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V=\pi r^2 h$.)

$dV/dt=-5\pi\sqrt{h}$

(a) Show that $\frac{dh}{dt}=-\frac{\sqrt{h}}{5}$.

(b) Given that $h=17$ at time $t=0$, solve the differential equation $\frac{dh}{dt}=-\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

A) $V=\pi r^2 h$

$\frac{dV}{dt}=\pi\left[2r\cancel{\frac{dr}{dt}h}+r^2\frac{dh}{dt}\right]$

$\frac{dV}{dt}=\pi r^2\frac{dh}{dt}$

$-5\pi\sqrt{h}=\pi(5)^2\frac{dh}{dt}$

$\frac{-5\pi\sqrt{h}}{25\pi}=dh/dt$

$=\frac{-\sqrt{h}}{5}=\frac{dh}{dt}$ ✓

B.) $\frac{dh}{dt}=\frac{-\sqrt{h}}{5}$

$5dh=-\sqrt{h}\,dt$

$\frac{-1}{\sqrt{h}}dh=\frac{1}{5}dt$

$\int -h^{-1/2}dh=\int\frac{1}{5}dt$

$-2h^{1/2}=\frac{1}{5}t+C$

$-2\sqrt{17}=\frac{1}{5}(0)+C$

$-2\sqrt{17}=C$

$-2\sqrt{h}=\frac{1}{5}t-2\sqrt{17}$

$\sqrt{h}=\frac{-1}{10}t+\sqrt{17}$

$\boxed{h=(-1/10\,t+\sqrt{17})^2}$

C) when $h=0$, coffee pot is empty!

$h=(-1/10\,t+\sqrt{17})^2=0$

$-1/10\,t=-\sqrt{17}$

$\boxed{t=10\sqrt{17}}$

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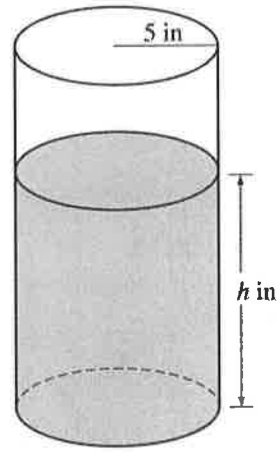
Question 5

A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

$3 : \begin{cases} 1 : \dfrac{dV}{dt} = -5\pi\sqrt{h} \\ 1 : \text{computes } \dfrac{dV}{dt} \\ 1 : \text{shows result} \end{cases}$

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{5}\,dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } h = 17 \\ \quad \text{when } t = 0 \\ 1 : \text{solves for } h \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

2008 AP Test Free Response Question #3

CALCULATOR

$dv/dt = 2000\,cm^3$

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

A) $r = 100\,cm,\ h = 0.5\,cm$

$dr/dt = 2.5\,cm/min$, whats dh/dt?

$V = \pi r^2 h$

$\frac{dv}{dt} = \pi\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$

$2000 = \pi\left(2(100)(2.5)(0.5) + (100)^2\frac{dh}{dt}\right)$

$\frac{2000}{\pi} = \left(250 + 10000\frac{dh}{dt}\right)$ use calc!!

$\boxed{dh/dt = 0.038\,cm/min}$

(the change in oil volume equals leaking rate minus removal rate.)

B.) $\frac{dv}{dt} = 2000 - R(t)$. Therefore $\frac{dv}{dt} = 0$ when $R(t) = 2000$

If $R(t) = 400\sqrt{t} = 2000$

$\sqrt{t} = 5$

$t = 25$ minutes

dv/dt: $+$ before 25, $-$ after 25

since dv/dt changes from pos. to neg. at t = 25 minutes its at a max volume @ 25 min!

C.) starting

$60000 + \int_0^{25} (2000 - R(t))\,dt$

Question 3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

(a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5$ cm/min.

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm/min}$$

$4: \begin{cases} 1: \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2: \text{expression for } \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$

(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$.
This occurs when $t = 25$ minutes.
Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working.

$3: \begin{cases} 1: R(t) = 2000 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) The volume of oil, in cm^3, in the slick at time $t = 25$ minutes is given by $60{,}000 + \int_0^{25} (2000 - R(t))\, dt$.

$2: \begin{cases} 1: \text{limits and initial condition} \\ 1: \text{integrand} \end{cases}$