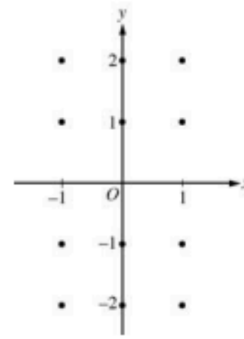
Question 6

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

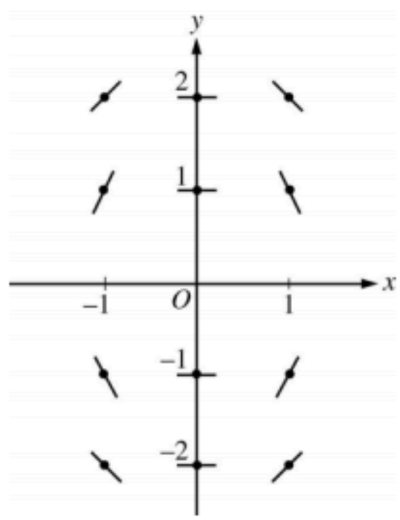
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the pink test booklet.)

(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.



(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \end{cases}$

(b) The line tangent to f at $(1, -1)$ is $y + 1 = 2(x - 1)$.
Thus, $f(1.1)$ is approximately -0.8.

$2 : \begin{cases} 1 : \text{equation of the tangent line} \\ 1 : \text{approximation for } f(1.1) \end{cases}$

(c) $\frac{dy}{dx} = -\frac{2x}{y}$

$y\,dy = -2x\,dx$

$\frac{y^2}{2} = -x^2 + C$

$\frac{1}{2} = -1 + C;\ C = \frac{3}{2}$

$y^2 = -2x^2 + 3$

Since the particular solution goes through $(1, -1)$, y must be negative.

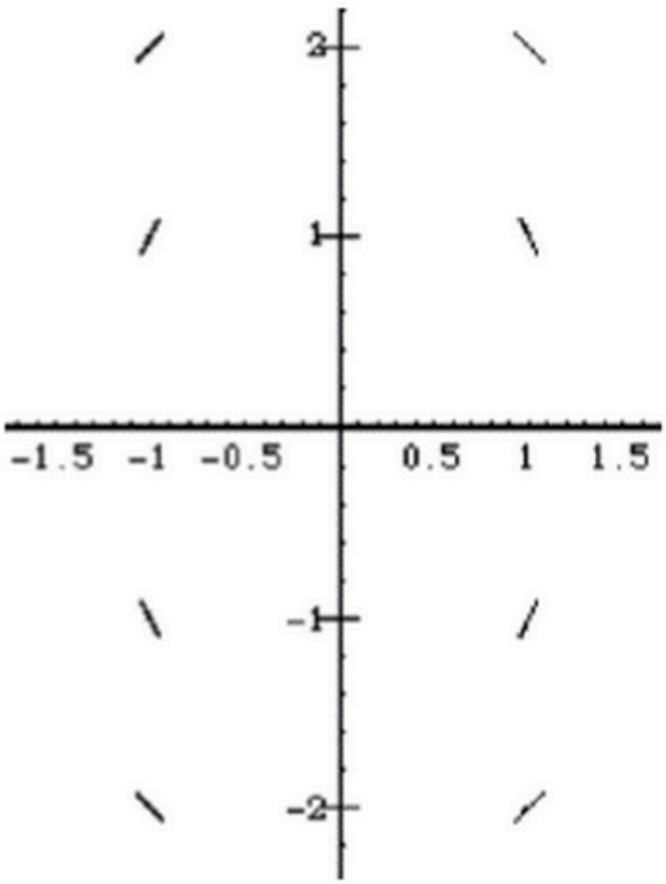
Thus the particular solution is $y = -\sqrt{3 - 2x^2}$.

$5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $2/5$ [1-1-0-0-0] if no constant of integration

Note: $0/5$ if no separation of variables

Work for 2005 Question #6



a.

Slopes	Row 1	1	0	–1
	Row 2	2	0	–2
	Row 3	–2	0	2
	Row 4	–1	0	1

b. Slope at (1,–1) is $-\dfrac{2(1)}{-1} = 2$. So equation of tangent line is (y + 1) = 2(x – 1).

So using this to estimate y with x = 1.1 → gives y + 1 = 2(1.1 – 1) or y = .2 – 1 $\approx \boxed{-.8}$

c. $\int y\,dy = \int -2x\,dx \rightarrow \dfrac{y^2}{2} = -x^2 + C$ Using initial condition (1,–1) gives

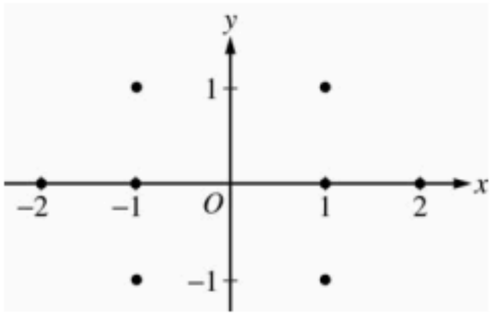
$\dfrac{(-1)^2}{2} = -(1)^2 + C \;\text{---}> C = \dfrac{3}{2}$ So $\dfrac{y^2}{2} = -x^2 + \dfrac{3}{2} \;\text{---}> y^2 = -2x^2 + 3$

Simplifying gives $y = \pm\sqrt{3 - 2x^2}$ For the initial condition to be true, namely that $f(x) = -1$, take the negative root. $\boxed{y = -\sqrt{3 - 2x^2}}$

Question 5

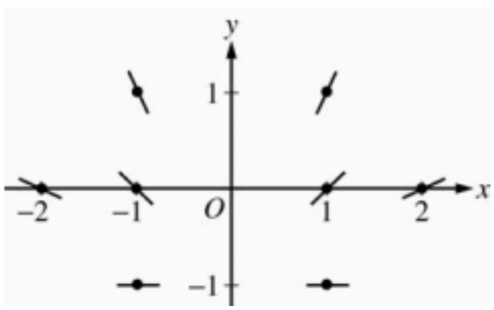
Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $x \neq 0$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.
(Note: Use the axes provided in the pink exam booklet.)



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

(a)



2 : sign of slope at each point and relative steepness of slope lines in rows and columns

(b) $\frac{1}{1+y}\,dy = \frac{1}{x}\,dx$

$\ln|1+y| = \ln|x| + K$

$|1+y| = e^{\ln|x| + K}$

$1 + y = C|x|$

$2 = C$

$1 + y = 2|x|$

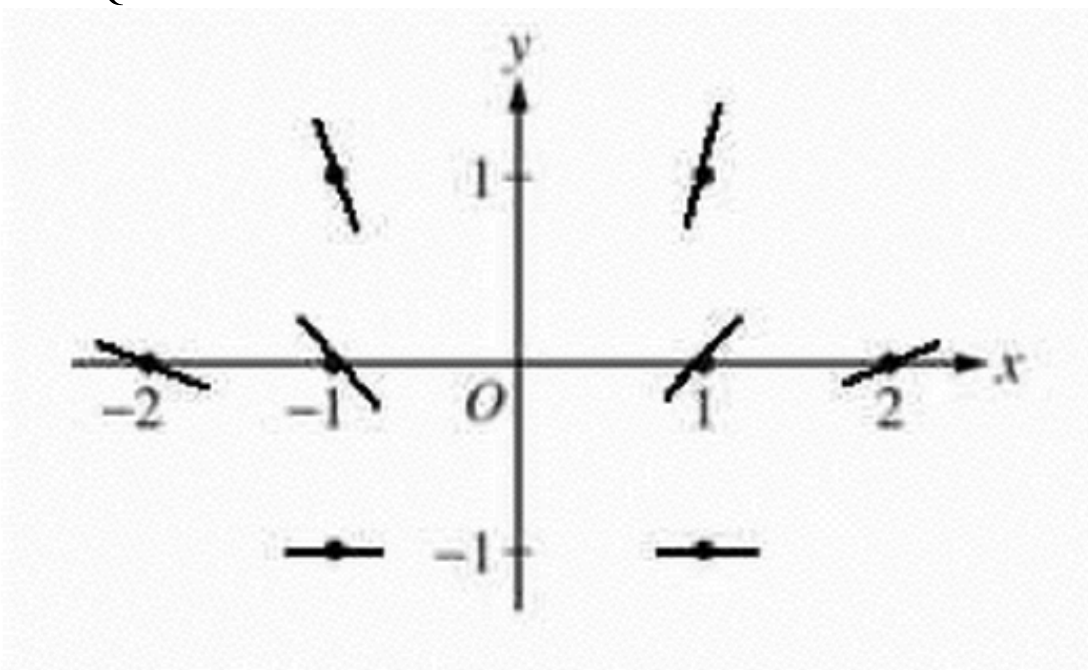
$y = 2|x| - 1$ and $x < 0$

or

$y = -2x - 1$ and $x < 0$

7 :
- 6 :
  - 1 : separates variables
  - 2 : antiderivatives
  - 1 : constant of integration
  - 1 : uses initial condition
  - 1 : solves for y
- Note: max $3/6$ [1-2-0-0-0] if no constant of integration
- Note: $0/6$ if no separation of variables
- 1 : domain

Work for 2006 Question #5



a.

Point	Slope	Point	Slope
(–2,0) = –½	(2,0) = ½	(–1,1) = –2	(1,1) = 2
(–1,0) = –1	(1,0) = 1	(–1,–1) = 0	(1,–1) = 0

b. Using variables separable,

$\int \frac{dy}{1+y} = \int \frac{dx}{x} \rightarrow \ln|1+y| = \ln|x| + C \rightarrow \ln|1+y| = \ln|x| + C \rightarrow$

$\ln 2 = \ln 1 + C \rightarrow C = \ln 2 \rightarrow \ln|1+y| = \ln|x| + \ln 2 \rightarrow$

$\ln|1+y| = \ln|2x| \rightarrow e^{\ln|1+y|} = e^{\ln|2x|} \rightarrow |y+1| = |2x|$. Since this can also be expressed as y + 1= ±2x, use the intial condition to determine which sign to use. At the point given, (–1,1) this equation would have to use the – sign to be true, so

$y+1 = -2x \rightarrow \boxed{y = -2x - 1}$. Since we are given that $\frac{dy}{dx}$ undefined at x = 0, then the

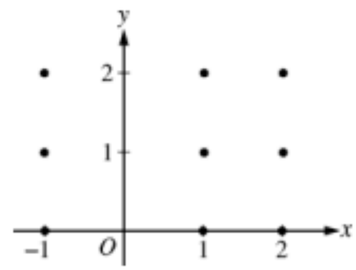
$\boxed{domain\ \ for\ \ x\ \ is\ (-\infty,0)\ \ or\ \ x<0}$.

AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

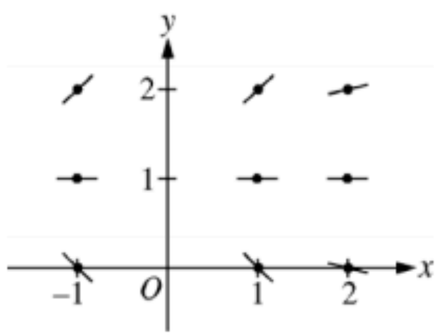
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \to \infty} f(x)$.

(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) $\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx$

$\ln|y-1| = -\frac{1}{x} + C$

$|y-1| = e^{-\frac{1}{x}+C}$

$|y-1| = e^C e^{-\frac{1}{x}}$

$y - 1 = ke^{-\frac{1}{x}}$, where $k = \pm e^C$

$-1 = ke^{-\frac{1}{2}}$

$k = -e^{\frac{1}{2}}$

$f(x) = 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)}$, $x > 0$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

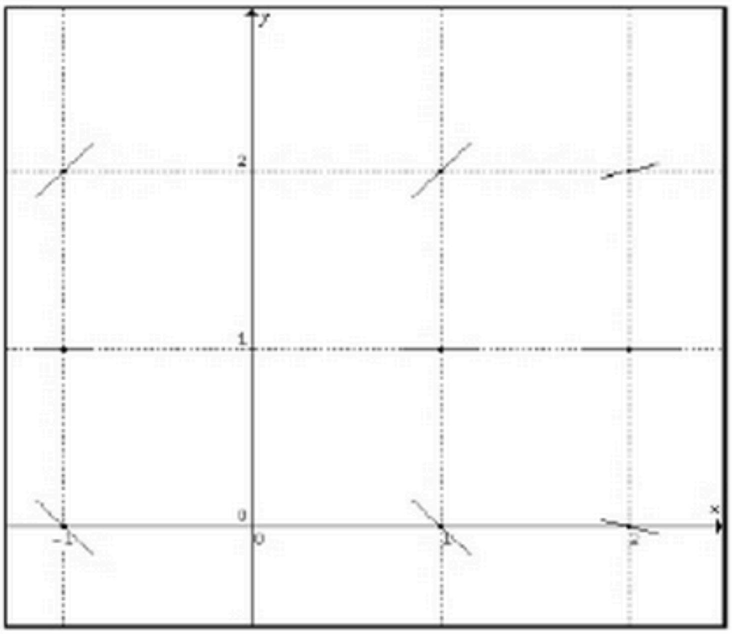
(c) $\lim_{x \to \infty} 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

Work for 2008 Question #5—**NOTE: This was your NON-CALCULATOR TEST QUESTION!!!!**

(a)

x	-1	-1	-1	1	1	1	2	2	2
y	0	1	2	0	1	2	0	1	2
dy/dx	-1	0	1	-1	0	1	-1/4	0	1/4



(b) $f(2)=0$ and $\dfrac{dy}{dx}=\dfrac{y-1}{x^2}$. Separating the variables, $\dfrac{dy}{y-1}=\dfrac{dx}{x^2}$.

Integrating, $\ln|y-1|=-\dfrac{1}{x}+C$. Since $f(2)=0$, $0=-\dfrac{1}{2}+C$, so $C=\dfrac{1}{2}$.

Then, $\ln|y-1|=-\dfrac{1}{x}+\dfrac{1}{2}$.

Since $f(2)=0<1$, then $\ln(1-y)=-\dfrac{1}{x}+\dfrac{1}{2} \rightarrow 1-y=e^{-\frac{1}{x}+\frac{1}{2}} \rightarrow \boxed{y=1-e^{-\frac{1}{x}+\frac{1}{2}}}$

(c) $\lim\limits_{x\to\infty}\left(1-e^{-\frac{1}{x}+\frac{1}{2}}\right) = 1-e^{0+1/2}=\boxed{1-e^{1/2}}$